PHYSICAL REVIEW E **70**, 035102(R) (2004)

Universality of the optimal path in the strong disorder limit

Sergey V. Buldyrev,[1,*] Shlomo Havlin,[1,2] Eduardo López,[1] and H. Eugene Stanley[1]
[1]*Center for Polymer Studies and Department of Physics, Boston University, Boston, Massachusetts 02215, USA*
[2]*Minerva Center and Department of Physics, Bar-Ilan University, 52900 Ramat-Gan, Israel*
(Received 10 February 2004; published 14 September 2004)

We study numerically the optimal paths in two and three dimensions on various disordered lattices in the limit of strong disorder. We find that the length ℓ of the optimal path scales with geometric distance r, as $\ell \sim r^{d_{\rm opt}}$ with $d_{\rm opt}=1.22\pm0.01$ for $d=2$ and 1.44 ± 0.02 for $d=3$, independent of whether the optimization is on a path of weighted bonds or sites, and independent of the lattice or its coordination number. Our finding suggests that the exponent $d_{\rm opt}$ is universal, depending only on the dimension of the system.

DOI: 10.1103/PhysRevE.70.035102 PACS number(s): 89.75.Hc

The statistical properties of optimal paths in a disordered energy landscape have been studied extensively in recent years [1–12]. Optimal path properties are relevant to many optimization problems, including the folding of proteins, spin glasses, and the well-known traveling salesman problem. Several studies considered the optimal paths in the strong disorder limit, where a single site or bond weight dominates the weight of the whole path, and found that the length ℓ of the optimal path scales with distance r as $r^{d_{\rm opt}}$, where $d_{\rm opt}=1.22\pm0.02$ in $d=2$ and $d_{\rm opt}=1.43\pm0.03$ in $d=3$ [6,10]. Since the optimal path can be mapped to the minimum spanning tree [13] and to invasion percolation with trapping [7,14], it follows that the paths between two sites separated at a distance r in all three problems scale as the optimal path, with the same exponents $d_{\rm opt}$.

Recently, Knackstedt *et al.* [15] studied invasion percolation with trapping and concluded that the scaling exponent of the minimal path depends on the lattice coordination number and is therefore not universal. They further argue that since optimal paths in strongly disordered media and minimum spanning trees on random graphs are related to invasion percolation, it follows that these problems also do not possess universal scaling properties. Here we directly study the optimal paths in the strong disorder limit and find that their scaling properties are universal. The fact that the scaling of optimization paths is universal enables one to study only one type of lattice for each dimension.

We perform numerical simulations in the strong disorder limit of the optimal path between two sites A and B in several two-dimensional (2D) and three-dimensional (3D) lattices with periodic boundaries. Strong disorder is usually implemented by assigning (to either sites or bonds of the lattice) random energies ϵ_i, uniformly distributed on an interval [0, 1], and computing the weights associated with them,

$$\tau_i \equiv \exp(\beta\epsilon_i), \tag{1}$$

where β is the strength of disorder which has the physical meaning of inverse temperature. The optimal path is the path connecting sites A and B, which minimizes the sum of weights of all visited sites or bonds on the way from A to B. The limit $\beta\to\infty$ is the strong disorder limit, where only the largest τ_i along the path dominates the sum. It is rigorously proved [6,10–12] that the optimization in strong disorder is equivalent to removing sites or bonds in random order, provided that the connectivity between A and B is not destroyed. This can be understood if the order of removal is determined by the descending values of energies of the sites or bonds.

For each lattice of size $L\times L$ in two dimensions or $L\times L\times L$ in three dimensions, we generate $M=10^4$ realizations of disorder implemented by the order of removal of sites or bonds. In all realizations, we place sites $A=(0,0,0)$ and $B=(r,0,0)$ at the same locations separated by distance $r\equiv L/2$. For each realization we compute the length of the optimal path, ℓ, left after removing all sites or bonds from the lattice, except those whose removal would destroy connectivity between A and B. In both cases (sites or bonds) the length of the path ℓ is defined as the number of bonds connecting sites A and B. We compute the distribution $P(\ell,r)$, the average $\ell_{\rm opt}=\langle\ell\rangle$, and the average square $\langle\ell^2\rangle$ over all realizations of disorder.

To implement various lattices with different coordination numbers, we always start with a square lattice in two dimensions or a cubic lattice in three dimensions. In the square lattice a site (i,j) is connected with four sites $(i,j\pm1)$, $(i\pm1,j)$. In the triangular lattice, of coordination number $z=6$, it is connected with two additional sites $(i+1,j+1)$ and $(i-1,j-1)$. In the "star" lattice with $z=8$ it is connected with two more sites $(i-1,j+1)$ and $(i+1,j-1)$. In the hexagonal lattice ($z=3$), every site (i,j) is connected with sites $(i-1,j)$ and $(i+1,j)$, and in addition, it is connected with sites $(i,j+1)$ if $i+j$ is even, or with $(i,j-1)$ if $i+j$ is odd.

In a simple cubic lattice each site (i,j,k) is linked with $z=6$ sites: $(i\pm1,j,k)$, $(i,j\pm1,k)$, and $(i,j,k\pm1)$. To implement a face-centered-cubic (fcc) lattice with coordination number 12 we connect each site (i,j,k) with sites $(i+1,j,k)$, $(i-1,j,k)$, $(i,j+1,k)$, $(i,j-1,k)$ in the same plane, with sites $(i,j,k+1)$, $(i+1,j,k+1)$, $(i,j+1,k+1)$, $(i+1,j+1,k+1)$ in the plane above, and with sites $(i,j,k-1)$, $(i-1,j,k-1)$, $(i,j-1,k-1)$, $(i-1,j-1,k-1)$, in the plane below.

We find that for both site and bond lattices in two and

*Current address: Physics Department, Yeshiva University, 500 W. 185 Street, New York, NY 10033.

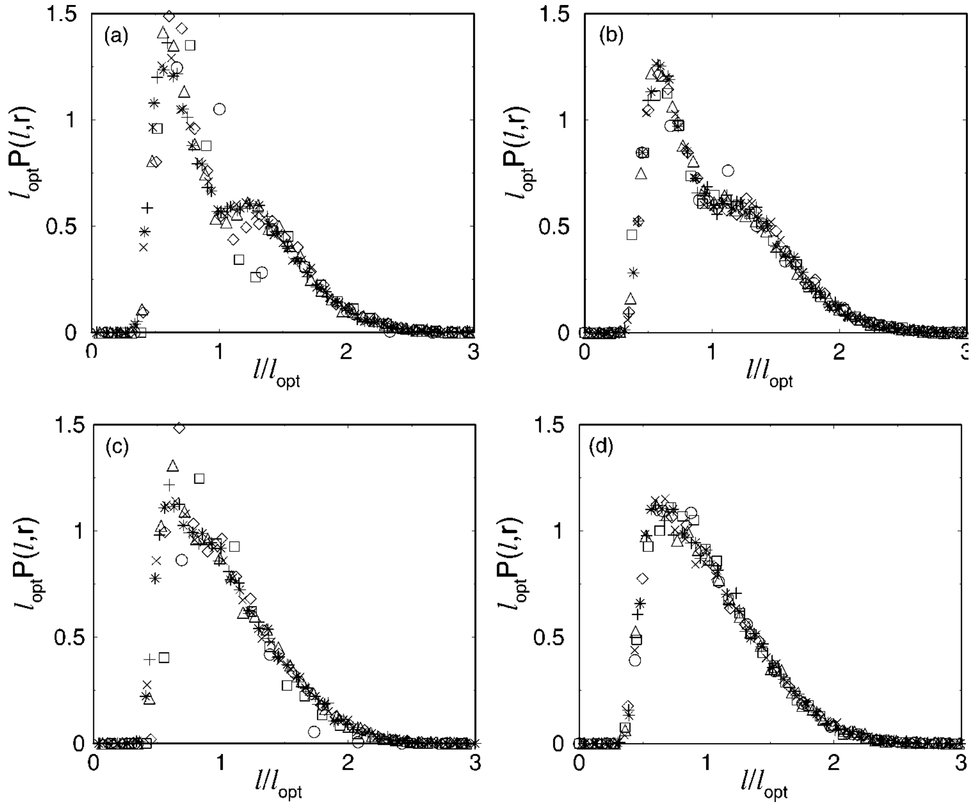

FIG. 1. Scaled distributions $P(\ell,r)$ for (a) site square lattice, (b) bond square lattice, (c) site triangular lattice, and (d) bond triangular lattice, for $r=2(\bigcirc)$, $r=4(\square)$, $r=8(\diamond)$, $r=16(\triangle)$, $r=32(+)$, $r=64(\times)$, and $r=128(*)$.

three dimensions, the distributions $P(\ell,r)$ converge for large r, as expected for a mass distribution of a fractal object to the functional form,

$$P(\ell,r) = \frac{1}{\ell_{opt}(r)} F\left[\frac{\ell}{\ell_{opt}(r)}\right], \tag{2}$$

where $F(x)$ is a function of a scaling variable $x \equiv \ell/\ell_{opt}$ (Fig. 1). The shape of the function $F(x) \equiv \lim_{r\to\infty} \ell_{opt} P(x\ell_{opt}, r)$ is caused by the particular geometry of the system with periodic boundaries. For example, the second peak of the distribution in the case of the square lattice is formed by the paths connecting A and B along the diagonal of the system. The sharp fall in the tails $x \gg 1$ is due to the effect of the boundaries, since $r=L/2$.

In analogy with the behavior of the distribution of the shortest path length on the percolation cluster [16,17], one can expect for $y \equiv L/r \gg 1$ and $1 \ll x \ll y^{d_{opt}}$ a power-law decay,

$$F(x) \sim x^{-g} f_1(x) f_2(x/y^{d_{opt}}), \tag{3}$$

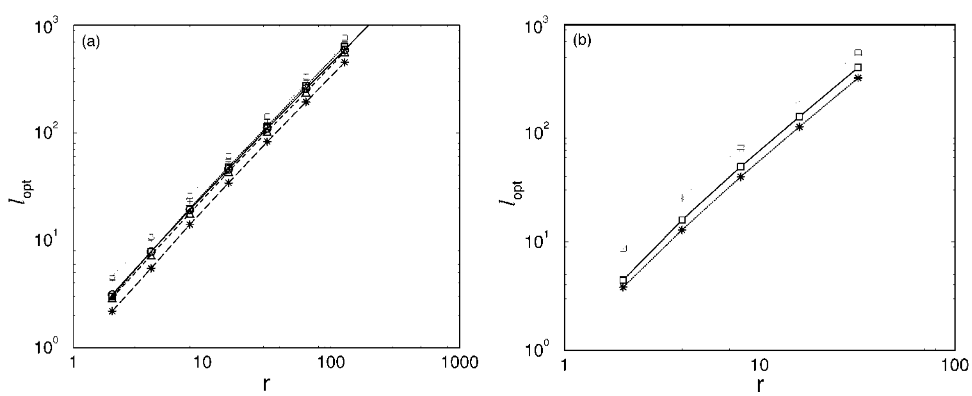

FIG. 2. (a) The dependence of ℓ_{opt} on r for hexagonal ($z=3$, solid line, $\bigcirc$), square ($z=4$, dotted line, $\square$), triangular ($z=6$, dashed line, $\triangle$), and star ($z=8$, long dashed line, $*$) lattices for the strong disorder implemented on sites (bold lines and symbols) and bonds (thin lines and symbols). (b) Same for cubic ($z=6$, solid line, $\square$) and fcc ($z=12$, dotted line, $*$) lattices, for sites (bold lines and symbols), and bonds (thin lines and symbols).

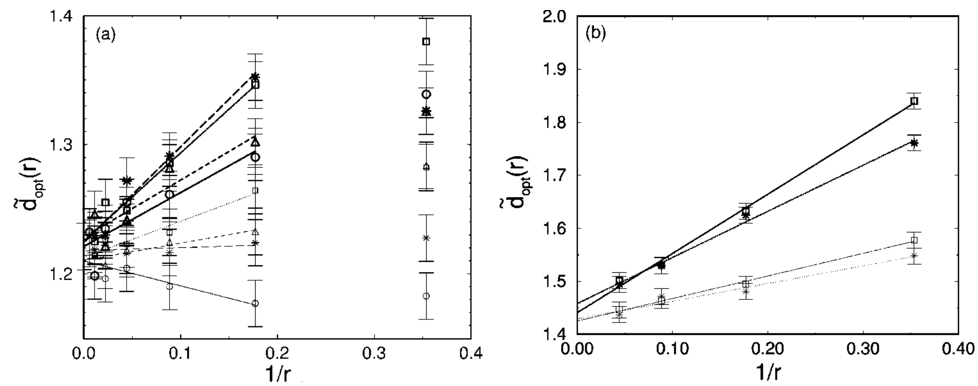

FIG. 3. (a) The dependence of successive slopes of $\tilde{d}_{opt}(r)$ on $1/r$ for hexagonal ($z=3$), square ($z=4$), triangular ($z=6$), and star ($z=8$) lattices for the strong disorder implemented on sites and bonds. (b) Same for cubic ($z=6$) and fcc ($z=12$) lattices. The symbols are the same as in Fig. 2. The linear fits are indicated by the same line styles as in Fig. 2.

where g is a universal exponent and $f_1(x)$ and $f_2(x)$ are lower and upper cutoff functions, respectively. The functional forms of the cutoff functions for the shortest path on a percolation cluster are given by

$$f_1(x) \sim \exp(-a_1 x^{-\delta_1}),$$

$$f_2(x) \sim \exp(-a_2 x^{\delta_2}), \tag{4}$$

where δ_1, δ_2 are positive scaling exponents and a_1, a_2 are positive lattice-dependent constants. Our numerical analysis shows that the same functional form holds for the distribution of the optimal path in strong disorder. In analogy with self-avoiding walks problem [18] we can conjecture that

$$\delta_1 = \frac{1}{d_{opt}-1}. \tag{5}$$

Plotting $\ln\{\ln[b/F(x)]\}$ vs $\ln(x)$, where b is a constant that must be selected to achieve the best straight-line fits, we find $\delta_1=5\pm1$ for all 2D lattices and $\delta_1=2.6\pm0.5$ for all 3D lattices, which is consistent with Eq. (5). We find also $\delta_2=2.5\pm0.5$ in two dimensions and $\delta_2=3.0\pm0.5$ in three dimensions. The values of exponent $g=1.6\pm0.1$ in two dimensions and $g=1.3\pm0.1$ in three dimensions can be found by simulating systems with large $y\gg1$ [19]. For $y=2$, which we study here, the power-law regime due to x^{-g} in Eq. (3) does not exist, and the sharp fall in the tail for $x\gg1$ in Fig. 1 is described by function $f_2(x)$. To find the exponent d_{opt} defined by the scaling relation $\ell_{opt}\sim r^{d_{opt}}$, we plot ℓ_{opt} vs r in a double logarithmic scale (Fig. 2), find its successive slopes $\tilde{d}_{opt}(r)$ of the data points, defined as

$$\tilde{d}_{opt}(r) \equiv \frac{\ln \ell_{opt}(r\sqrt{2}) - \ln \ell_{opt}(r/\sqrt{2})}{\ln 2}, \tag{6}$$

and plot them vs $1/r$ (Fig. 3). The error bars for each point are estimated to be $2\sigma(r)/(\sqrt{M}\ln 2)$, where $\sigma(r)$ is the relative standard deviation of the distribution $P(\ell,r)$,

$$\sigma(r) \equiv \frac{\sqrt{\langle \ell^2\rangle - \langle \ell\rangle^2}}{\langle \ell\rangle}. \tag{7}$$

Note, that due to Eq. (2), $\sigma(r)\to\sigma_0$, where σ_0 is the standard deviation of $F(x)$ (Fig. 4). Thus, the errors in $\tilde{d}_{opt}(r)$ practically do not depend on r and constitute for $M=10^4$ less than a percent.

We determine the value of d_{opt} for each lattice as the value of the y intercept of the least-square linear fit (Fig. 3). This fit assumes corrections to scaling of the form

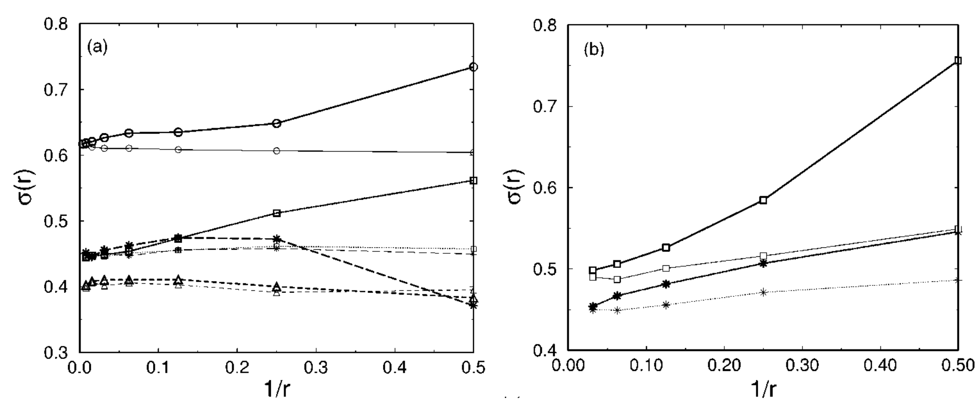

FIG. 4. The relative standard deviation $\sigma(r)$ of the distribution $P(\ell,r)$ as function of $1/r$ in (a) two and (b) three dimensions. The symbols and line styles are the same as in Fig. 2.

TABLE I. The values of the scaling exponent d_{opt} for different lattices of various dimensionality d and coordination number z for strong disorder implemented on sites and bonds.

Lattice type	d	z	d_{opt} (site disorder)	d_{opt} (bond disorder)
Hexagonal	2	3	1.221 ± 0.02	1.210 ± 0.02
Square	2	4	1.226 ± 0.02	1.213 ± 0.02
Triangular	2	6	1.228 ± 0.02	1.210 ± 0.02
Star	2	8	1.224 ± 0.02	1.218 ± 0.02
Cubic	3	6	1.441 ± 0.03	1.425 ± 0.03
fcc	3	12	1.458 ± 0.03	1.429 ± 0.03

$$\ell_{opt} = a(r+b)^{d_{opt}} + o(r^{d_{opt}-1}), \tag{8}$$

where a and b are lattice-dependent constants. The values of d_{opt} for each lattice are presented in Table I. All these values are within the error bars from

$$d_{opt} = \begin{cases} 1.22 \pm 0.01 & (2D) \\ 1.44 \pm 0.02 & (3D). \end{cases} \tag{9}$$

This result for $d=2$ is quite different from the value of 1.135 ± 0.003 obtained in Ref. [15] for the shortest path in invasion site percolation with trapping in triangular and star lattices [20]. Our result $d_{opt}=1.22\pm0.01$ is consistent with Ref. [15] for other site and bond lattices.

In summary, we find that the values of d_{opt} are universal for all lattice types studied for both site and bond problems, and depends only on the dimensionality of lattice d. These findings agree with the assumption that d_{opt} monotonically increases with d from $d_{opt}/d=1$ for $d=1$, to $d_{opt}=2$ for $d=d_c=6$, which is the upper critical dimension of percolation [21], since for $d \geqslant d_c$, we expect to recover for ℓ_{opt} a random-walk behavior with $d_{opt}=2$. Since a random network corresponds to an infinite-dimensional lattice, the latter value defines the behavior of ℓ_{opt} as a function of the number of sites N on a random network,

$$\ell_{opt} \sim r^{d_{opt}} \sim N^{d_{opt}/d_c} \sim N^{1/3}. \tag{10}$$

The scaling (10) was found to hold for random networks for both site and bond disorder, and any coordination number [12]. Since invasion percolation with trapping and minimal spanning trees are mapped to optimization in strong disorder, our results suggest also that these systems possess universal character—in contrast with the conclusion in Ref. [15].

We wish to thank P. Grassberger for discussions and the Israel Science Foundation and the Office of Naval Research for support.

[1] D. A. Huse and C. L. Henley, Phys. Rev. Lett. **54**, 2708 (1985); M. Kardar, *ibid.* **55**, 2923 (1985); D. A. Huse, C. L. Henley, and D. S. Fisher, *ibid.* **55**, 2924 (1985).
[2] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986); M. Kardar and Y.-C. Zhang, *ibid.* **58**, 2087 (1987).
[3] E. Perlsman and M. Schwartz, Europhys. Lett. **17**, 11 (1992); E. Perlsman, M. Schwartz, and S. F. Edwards, *ibid.* **20**, 301 (1992); E. Perlsman and M. Schwartz, Physica A **234**, 523 (1996).
[4] T. Halpin-Healy and Y.-C. Zhang, Phys. Rep. **254**, 215 (1995).
[5] A.-L. Barabási and H. E. Stanley, *Fractal Concepts in Surface Growth* (Cambridge University Press, Cambridge, 1995).
[6] M. Cieplak, A. Maritan, and J. R. Banavar, Phys. Rev. Lett. **72**, 2320 (1984) ; **76**, 3754 (1996).
[7] A.-L. Barabási, Phys. Rev. Lett. **76**, 3750 (1996).
[8] P. De Los Rios and Y.-C. Zhang, Phys. Rev. Lett. **81**, 1023 (1998).
[9] N. Schwartz, A. L. Nazaryev, and S. Havlin, Phys. Rev. E **58**, 7642 (1998).
[10] M. Porto *et al.*, Phys. Rev. E **60**, R2448 (1999).
[11] L. A. Braunstein, S. V. Buldyrev, S. Havlin, and H. E. Stanley, Phys. Rev. E **65**, 056128 (2002).
[12] L. A. Braunstein, S. V. Buldyrev, R. Cohen, S. Havlin, and H. E. Stanley, Phys. Rev. Lett. **91**, 168701 (2003).
[13] R. Dobrin and P. M. Duxbury, Phys. Rev. Lett. **86**, 5076 (2001).
[14] M. Porto, S. Havlin, S. Schwarzer, A. Bunde, Phys. Rev. Lett. **79**, 4060 (1997).
[15] M. A. Knackstedt, M. Sahimi, and A. P. Sheppard, Phys. Rev. E **65**, 035101(R) (2002).
[16] N. V. Dokholyan, Y. Lee, S. V. Buldyrev, S. Havlin, P. R. King, and H. E. Stanley, J. Stat. Phys. **93**, 603 (1998).
[17] J. S. Andrade Jr., S. V. Buldyrev, N. Dokholyan, P. R. King, Y. Lee, S. Havlin, and H. E. Stanley, Phys. Rev. E **62**, 8270 (2000).
[18] P. G. De Gennes, *Scaling Concepts in Polymer Physics* (Cornell University Press, Ithaca, 1979).
[19] E. López, Ph.D. thesis, Boston University, 2005.
[20] The value of 1.135 is actually very close to the scaling exponent for the shortest path on a regular percolation cluster in two dimensions, see P. Grassberger, Physica A **262**, 251 (1999).
[21] S. Redner, Phys. Rev. B **25**, 3242 (1982).